

Process control

Lecture 6

Laplace domain

The use of Laplace transformation is very useful in the analysis of dynamics and in the control system theory. It is an integral transformation defined by the equation:

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt = F(s) \quad (1)$$

where t is time, \mathcal{L} is Laplace transform operator and s is a complex variable. A function of complex variable $F(s)$ is obtained as a result of the transformation. For example, if $f(t) = 1$, then, the Laplace transform of this function is:

$$\mathcal{L}[1] = \int_0^{\infty} 1 \cdot e^{-st} dt = \left[-\frac{1}{s} e^{-st} \right]_{t=0}^{t \rightarrow \infty} = \left[0 + \frac{1}{s} \right] = \frac{1}{s} \quad (2)$$

For the function $f(t) = Kt$, we have:

$$\mathcal{L}[Kt] = \int_0^{\infty} K \cdot t \cdot e^{-st} dt = K \int_0^{\infty} t \cdot e^{-st} dt \quad (3)$$

Integrating by parts:

$$\begin{aligned} K \int_0^{\infty} t \cdot e^{-st} dt &= \left. \begin{array}{l} u = t \quad v' = e^{-st} \\ u' = 1 \quad v = -\frac{1}{s} e^{-st} \end{array} \right| = \left[-\frac{Kt}{s} e^{-st} \right]_{t=0}^{t \rightarrow \infty} - K \int_0^{\infty} -\frac{1}{s} e^{-st} dt = \\ &= \left[-\frac{Kt}{s} e^{-st} \right]_{t=0}^{t \rightarrow \infty} + \frac{K}{s} \int_0^{\infty} e^{-st} dt = \left[-\frac{Kt}{s} e^{-st} \right]_{t=0}^{t \rightarrow \infty} - \left[\frac{K}{s^2} e^{-st} \right]_{t=0}^{t \rightarrow \infty} = \\ &[0 - 0] - \frac{K}{s^2} = \frac{K}{s^2} \end{aligned} \quad (4)$$

It is necessary to know the properties of Laplace transform. One of them is linearity:

$$\mathcal{L}[f_1(t) + f_2(t)] = \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)] = F_1(s) + F_2(s) \quad (5)$$

It is also easy to prove that:

$$\mathcal{L}[a \cdot f(t)] = a\mathcal{L}[f(t)] \quad (6)$$

Another important property is the Laplace transform of function derivative:

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = s\mathcal{L}[f(t)] - f(0) \quad (7)$$

and second-order derivative:

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s^2\mathcal{L}[f(t)] - sf(0) - \frac{df(0)}{dt} \quad (8)$$

In order to transform the function of a complex variable s , an invert Laplace transform is used. It is calculated as follows:

$$f(t) = \frac{1}{2\pi i} \int_{\alpha-i\omega}^{\alpha+i\omega} e^{st} F(s) ds \quad (9)$$

Commonly in practice, inverse Laplace transform is found using partial fraction expansion.

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[F_1(s) + F_2(s) + \dots + F_n(s)] = \mathcal{L}^{-1}[F_1(s)] + \mathcal{L}^{-1}[F_2(s)] + \dots + \mathcal{L}^{-1}[F_n(s)] \quad (10)$$

Above way makes it simpler to find the inverse Laplace transform, because the problem is changed to several simpler ones. Because calculating inverse Laplace transform using the definition (9) is difficult it is often easier to look up the function in mathematic tables.

Example: Find the inverse Laplace transform of the function:

$$F(s) = \frac{K}{s(s+1/\tau)} \quad (11)$$

Using partial fraction expansion:

$$F(s) = \frac{K}{s(s+1/\tau)} = \frac{A}{s} + \frac{B}{s+1/\tau} \quad (12)$$

The roots of the denominator are 0 and $-1/\tau$.

$$A = \lim_{s \rightarrow 0} [sF(s)] = \lim_{s \rightarrow 0} \left[s \frac{K}{s(s+1/\tau)} \right] = K\tau$$

$$B = \lim_{s \rightarrow -1/\tau} [(s+1/\tau)F(s)] = \lim_{s \rightarrow -1/\tau} \left[(s+1/\tau) \frac{K}{s(s+1/\tau)} \right] = -K\tau$$

Therefore:

$$F(s) = \frac{K\tau}{s} - \frac{K\tau}{(s+1/\tau)} \quad (13)$$

More examples on partial fraction expansion can be found here:

<https://lpsa.swarthmore.edu/BackGround/PartialFraction/PartialFraction.html>

The inverse Laplace transform is:

$$\mathcal{L}[F(s)] = \mathcal{L}\left[\frac{K\tau}{s}\right] - \mathcal{L}\left[\frac{K\tau}{(s+1/\tau)}\right] = K\tau \mathcal{L}\left[\frac{1}{s}\right] - K\tau \mathcal{L}\left[\frac{1}{(s+1/\tau)}\right] \quad (14)$$

We will use mathematic tables (attached to the lecture), part of which is shown below.

Lp.	F(s)	f(t)
1	1	$\delta(t)$ - funkcja impulsowa
2	$\frac{a}{s}$	a
3	$\frac{1}{s^2}$	t
4	$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$ dla $n=1,2,3,\dots$
5	$\frac{1}{s+a}$	e^{-at}
6	$\frac{1}{1+s\tau}$	$\frac{1}{\tau} e^{-t/\tau}$
7	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1 - e^{-at})$

Fig. 1. Part of Laplace transform table

According to row 2, we know that $\mathcal{L}\left[\frac{1}{s}\right] = 1$ (a is equal 1) and according to row 5, that

$\mathcal{L}\left[\frac{1}{s+1/\tau}\right] = e^{-t/\tau}$ (a is equal $1/\tau$). Finally, we have:

$$f(t) = \mathcal{L}^{-1}[F(s)] = K\tau(1 + e^{-t/\tau}) \quad (15)$$

Laplace transform can be used for solving differential equations. The procedure is following:

- a) Transform the differential equation.
- b) Solve the algebraic equation with regard to $Y(s)$.
- c) Apply inverse Laplace transform to obtain $y(t)$.

Example: Solve the differential equation using the Laplace transform.

$$\frac{dy}{dt} - 1 = e^{-t} \quad y(0) = 1 \quad (16)$$

After applying the Laplace transform:

$$\mathcal{L}\left[\frac{dy}{dt} - 1\right] = \mathcal{L}[e^{-t}] \quad y(0) = 1 \quad (17)$$

After using the properties of Laplace transform:

$$\mathcal{L}\left[\frac{dy}{dt}\right] - \mathcal{L}[1] = \mathcal{L}[e^{-t}] \quad (18)$$

Using the property described by Eq. (7):

$$\begin{aligned} s\mathcal{L}[y] - y(0) - \mathcal{L}[1] &= \mathcal{L}[e^{-t}] \Rightarrow \\ s\mathcal{L}[y] - 1 - \mathcal{L}[1] &= \mathcal{L}[e^{-t}] \end{aligned} \quad (19)$$

Laplace transform of function $y(t)$ is not know, so we will denote it by $Y(s)$. After applying formulas from the table (rows 2 and 5):

$$sY(s) - 1 - \frac{1}{s} = \frac{1}{s+1} \quad (20)$$

Then, we solve the equation with regard to $Y(s)$:

$$Y(s) = \frac{1}{s(s+1)} + \frac{1}{s} + \frac{1}{s^2} \quad (21)$$

The function $y(t)$ is equal to inverse Laplace transform:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}\left[\frac{1}{s(s+1)} + \frac{1}{s} + \frac{1}{s^2}\right] = \\ &= \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \end{aligned} \quad (22)$$

The first term can be split by partial fraction expansion, but this is not necessary, as the adequate formula is available in the table (row no. 7, substitute $a = 1$). Finally, after using formulas in the table, we get:

$$y(t) = 1 - e^{-t} + 1 + t = 2 - e^{-t} + t \quad (23)$$